Problem Set 3

Risk-Averse Agent

Main Points

- Moral Hazard: Moral hazard occurs when the principal cannot observe or verify the agent's actions and the principal and agent have conflicting interests.
- Implications of Moral Hazard: When the principal cannot observe or verify the agent's actions, the contract cannot be based on these actions, but can only specify payment that may or may not be tied to the outcome.
- Optimal Contract: When the agent is risk averse, the principal is risk neutral, and the principal cannot observe or verify the agent's actions, it is optimal to tie the agent's pay to the outcome. The extent to which the pay is tied to the outcome depends negatively on the variation in the outcome due to factors that the agent cannot control and on the degree of agent's risk aversion.

Main Concepts

Hidden action; Moral hazard; First-best outcome; Observable actions; Verifiable actions; Incentive compatibility constraint.

Problems

(1) A hospital CEO considers employing an orthopaedic surgeon to perform knee surgeries. The number of knee surgeries the surgeon can perform depends stochastically on his effort according to $q=9e+u$, where u is a random variable with a mean of 0 and a variance of 4, and e is the surgeon's effort that cannot be observed by the hospital CEO. The surgeon's cost of effort is $c(e)=0.5e^2$ and his outside option is 20. The surgeon is risk-averse, with the coefficient of absolute risk aversion equal to 5. The hospital CEO is risk-neutral and her outside option is 0. If the hospital CEO were to offer a contract that includes a base salary plus a performance-based bonus (bq), what is the expected size of the bonus?

(2) A school principal considers hiring a new English teacher. For each unit of teacher's effort e, the students' standardized score on reading q increases by e+u, where u is a random variable with a mean of 0 and a variance of 1. The school principal cannot observe teacher's effort. The teacher's cost of effort is $c(e)=0.5e^2$ and his outside option is 0. The teacher is risk-averse, with the coefficient of absolute risk aversion equal to 3. The school principal is risk-neutral and her outside option is 0. If the school principal designs an optimal contract of the form $w=a+bq$, what variance in pay can the teacher expect?

(3) You consider hiring a Certified Financial Analyst (CFA) to manage your investment portfolio. The value of your portfolio depends stochastically on the analyst's effort according to $q=e+u$, where u is a random variable with a mean of 0 and a variance of 3, and e is the analyst's effort that you cannot observe. The analyst's cost of effort is

$c(e)=e^2$ and his outside option is 0. The analyst is risk-averse, with the coefficient of absolute risk aversion equal to 3. You are risk-neutral and your outside option is 0. What is the expected value of your portfolio if you designed an optimal contract of the form $w=a+bq$?

(4) For each unit of effort e, the sales associate sells $q=10e+u$ shirts, where u is a random variable with a mean of 0 and a variance of 3. The associate's cost of effort is $0.5e^2$, her outside option is 5, and her coefficient of absolute risk aversion is 2. The firm that employs the associate is risk neutral and its outside option is 0. Suppose that the firm offers a piece rate of the form $a+bq$, where a is the base salary and b is the commission rate. Compare the firm's and associate's expected payoffs when the firm can observe the associate's effort and when it cannot observe the associate's effort.

(5) You wish to hire an accountant to help you find legal savings in your tax return. For each unit of effort e, the accountant can increase your savings by $e+u$, where u is a random variable with a mean of 0 and a variance of 1. The accountant's cost of effort is $0.5e^2$, her outside option is 0, and her coefficient of absolute risk aversion is 2. You are risk neutral and your outside option is 0. Suppose that you offer the accountant a retainer (a) plus an additional payment (bq) that depends on the actual savings found. Compare the social surplus when you can observe the accountant's effort to the case when you cannot observe the accountant's effort.

(6) Consider a principal-agent relationship in which both the principal and the agent are risk averse. Specifically, the coefficients of absolute risk aversion are $s>0$ and $r>0$ for the principal and the agent, respectively. Suppose that the output depends stochastically on the agent's effort according to $q=e+u$, where u is a random variable with a mean of 0 and a variance of θ, and e is the agent's effort that cannot be observed by the principal. The agent's cost of effort is $c(e)=0.5e^2$ and his outside option is R. The principal's outside option is $S=0$. Find the optimal piece rate contract. Discuss how the optimal piece rate varies with each of θ, r, and s.

(7) Consider a principal-agent relationship in which there is no uncertainty. Specifically, suppose that the output depends on the agent's effort according to $q=e$, where e is the agent's effort that cannot be observed by the principal. Suppose further that the agent's cost of effort is $c(e)=0.5e^2$ and that the outside options are 0 for both the principal and the agent. Show that in this case, when there is hidden action but no uncertainty, there is no moral hazard problem: the principal can induce the optimal level of effort by paying $w=0.5$ if $q \geq q(e^*)$ and 0 otherwise.

(8) The economic historians Lee Alston and Robert Higgs analyzed three types of sharecropping contracts between land owners and farmers in the southern U.S. in the early 20th century: (1) wage labour, in which the land owner pays a fixed wage to the farmer in exchange for his labour; (2) land rental, in which the farmer pays a fixed rent to the land owner in exchange for the opportunity to work on land and harvest it; and (3) the sharecropping contract, in which the land owner and the farmer share the harvest. Alston and Higgs found that counties with greater crop risk made more use of wage labour and crop sharing relative to land rental. Is this finding consistent with economic theory?

Suggested Solutions

(These solutions are intended to be accurate and as complete as possible. Please report any remaining errors to jasmin.kantarevic@oma.org.)

(1) The expected size of the bonus is E[bq]=bE[q]=9be. Therefore, we need to find optimal b and e to calculate the expected bonus. We'll consider the problem of the CEO maximizing her expected payoff subject to the surgeon's individual rationality and participation constraints. We can start with the surgeon's expected payoff: $E[U]=E[w]-c(e)-RP^A=a+bE[q]-0.5rVar[w]=a+9be-0.5e^2-0.5rb^2\theta=a+9be-0.5e^2-10b^2$. The individual rationality constraint is then $\partial E[U]/\partial e=0$, or 9b-e=0, which implies that e=9b. The participation constraint is given by E[U]=R, from which we can express a as $a=R+c(e)+RP^A-bE[q]=20+0.5e^2+10b^2-9be$. Next, we can find the CEO's expected payoff: $E[V]=E[q-w]-RP^P=9e(1-b)-a$, since the CEO is risk neutral and therefore $RP^P=0$. Substituting the IR and PC into E[V], we get $E[V]=9e-0.5e^2-10b^2-20=9(9b)-0.5(9b)^2-10b^2-20=81b-40.5b^2-10b^2-20$. The first-order condition for b is then $\partial E[V]/\partial b=0$, or 81-81b-20b=0, which yields b≈0.8. Therefore, e=9b=7.2 and the expected bonus is E[bq]=9be≈51.8. To verify that this contract is also acceptable to the CEO, we have to check that E[V]≥S=0. Given b=0.8 and e=7.2, we have that $E[V]=81b-40.5b^2-10b^2-20=12.5>0$.

(2) The variance in teacher's pay is given by $Var[w]=Var[a+bq]=b^2\theta=b^2$ since the question assumes θ=1. Therefore, we need to find optimal b first. We'll consider the problem of the school principal maximizing her expected payoff subject to the teacher's individual rationality and participation constraints. We can start with the teacher's expected payoff: $E[U]=E[w]-c(e)-RP^A=a+bE[q]-0.5rVar[w]=a+be-0.5e^2-0.5(3)b^2(1)=a+be-0.5e^2-1.5b^2$. The individual rationality constraint is then $\partial E[U]/\partial e=0$, or b-e=0, which implies that e=b. The participation constraint is given by E[U]=R, from which we can express a as $a=R+c(e)+RP^A-bE[q]=0.5e^2+1.5b^2-be$. Next, we can find the CEO's expected payoff: $E[V]=E[q-w]-RP^P=e(1-b)-a$, since the CEO is risk neutral and therefore $RP^P=0$. Substituting the IR and PC into E[V], we get $E[V]=e-0.5e^2-1.5b^2=b-0.5b^2-1.5b^2$. The first-order condition for b is then $\partial E[V]/\partial b=0$, or 1-4b=0, which yields b=0.25. Therefore, the variance in teacher's pay is $Var[w]=b^2=(0.25^2)=0.0625$. To verify that this contract is also acceptable to the school principal, we have to check that E[V] ≥ R = 0. Given b=0.25, we have that $E[V]= b-0.5b^2-1.5b^2=0.125>0$.

(3) The expected value of your portfolio is E[q]=e. Therefore, we need to find optimal e. We'll consider the problem of maximizing your expected payoff subject to the analyst's individual rationality and participation constraints. We can start with the analyst's expected payoff: $E[U]=E[w]-c(e)-RP^A=a+bE[q]-0.5rVar[w]=a+be-e^2-0.5(3)b^2(3)=a+be-e^2-4.5b^2$. The individual rationality constraint is then $\partial E[U]/\partial e=0$, or b-2e=0, which implies that e=0.5b. The participation constraint is given by E[U]=R, from which we can express a as $a=R+c(e)+RP^A-bE[q]=e^2+4.5b^2-be$. Next, we can find your expected payoff: $E[V]=E[q-w]-RP^P=e(1-b)-a$, since you are risk neutral and therefore $RP^P=0$. Substituting the IR and PC into your expected payoff, we get $E[V]=e-e^2-4.5b^2=0.5b-(0.5b)^2-4.5b^2=0.5b-0.25b^2-4.5b^2$. The first-order condition for b is then $\partial E[V]/\partial b=0$, or 0.5-0.5b-9b=0, which yields b≈0.05. Therefore, the expected value of your portfolio is E[q]=e=0.5b=0.5(0.05)=0.025. To verify that this contract is also acceptable to you, we have to check that E[V]≥R=0. Given b=0.05, we have that $E[V]= 0.5b-0.25b^2-4.5b^2 \approx 0.013>0$. Therefore, this contract would be optimal to you and it would be efficient to hire the analyst.

(4) Consider first the case when the firm can observe the associate's effort. The optimal level of effort is given by $E[q'(e^*)]=c'(e^*)$, which implies that e*=10. In addition, given that the firm is risk-neutral while the associate is risk-averse, it is optimal that the firm completely insures the

associate, i.e. $w=a$ and $b=0$. From the participation constraint, we have that $E[U]=R$, or $a-c(e)=a-0.5e^2=a-0.5(10)^2=a-50=R=5$. Therefore, $a=55$. The firm's expected payoff is then $E[V]=E[q]-E[w]=10e-a=10(10)-55=45$. The associate's expected payoff is simply equal to his outside option $R=5=E[U]$. Consider next the case when the firm cannot observe the associate's effort. In this case, the associate's expected payoff is $E[U]=E[w]-c(e)-RP^A=a+bE[q]-0.5rVar[w]=a+10be-0.5e^2-0.5(2)b^2(6)=a+10be-0.5e^2-3b^2$. The individual rationality constraint is then $\partial E[U]/\partial e=0$, or $10b-e=0$, which implies that $e=10b$. The participation constraint is given by $E[U]=R$, from which we can express a as $a=R+c(e)+RP^A-bE[q]=5+0.5e^2+3b^2-10be$. Next, we can find the firm's expected payoff: $E[V]=E[q-w]-RP^P=10e(1-b)-a$, since the firm is risk neutral and therefore $RP^P=0$. Substituting the IR and PC into the firm's expected payoff, we get $E[V]=10e-0.5e^2-3b^2-5=10(10b)-0.5(10b)^2-3b^2-5=100b-50b^2-3b^2-5$. The first-order condition for b is then $\partial E[V]/\partial b=0$, or $100-100b-6b=0$, which yields $b\approx 0.94$. Therefore, the firm's expected payoff is $E[V]=100b-50b^2-3b^2-5\approx 42.2$. The associate's expected payoff is still 5 from the participation constraint. Therefore, the firm is strictly better off when there is no hidden action (45 relative to 42), while the associate is indifferent between the two cases. Notice also that the associate's effort is smaller when his action is hidden ($e=10b=10(0.94)=9.4$) compared to the case when the action is not hidden ($e^*=10$).

(5) The social surplus is given by $SS=E[V]+E[U]=E[q-w]-RP^P+E[w]-c(e)-RP^A=E[q]-c(e)-RP^A=e-0.5e^2-0.5(2)b^2(1)=e-0.5e^2-b^2$. Consider first the case when you can observe the accountant's effort. The optimal level of effort is given by $E[q'(e^*)]=c'(e^*)$, which implies that $e^*=1$. In addition, given that you are risk-neutral while the accountant is risk-averse, it is optimal that you completely insure the accountant, i.e. $b=0$. Therefore, the social surplus is $SS=e-0.5e^2-b^2=1-0.5(1^2)-(0^2)=0.5$. Consider next the case when you cannot observe the accountant's effort. The accountant's expected payoff is then $E[U]=E[w]-c(e)-RP^A=a+bE[q]-0.5rVar[w]=a+be-0.5e^2-0.5(2)b^2(1)=a+be-0.5e^2-b^2$. The individual rationality constraint is then $\partial E[U]/\partial e=0$, or $b-e=0$, which implies that $e=b$. The participation constraint is given by $E[U]=R$, from which we can express a as $a=R+c(e)+RP^A-bE[q]=0+0.5e^2+b^2-be$. Next, we can find your expected payoff: $E[V]=E[q-w]-RP^P=e(1-b)-a$, since you are risk neutral and therefore $RP^P=0$. Substituting the IR and PC into your expected payoff, we get $E[V]=e-0.5e^2-b^2=b-0.5b^2-b^2$. The first-order condition for b is then $\partial E[V]/\partial b=0$, or $1-b-2b=0$, which yields $b=1/3$ and $e=1/3$. Therefore, the social surplus is $SS=e-0.5e^2-b^2=(1/3)-0.5(1/3)^2-(1/3)^2=1/6\approx 0.17$. Therefore, the social surplus is smaller when the action is hidden (0.17) compared to the case when the action is not hidden (0.5).

(6) The expected payoff for the agent in this case is $E[U]=E[w]-c(e)-RP^A=a+be-0.5e^2-0.5rb^2\theta$. The individual rationality constraint is then $b=e$ and from the participation constraint, we have that $a=0.5e^2+0.5rb^2\theta$. The principal's expected payoff is then $E[V]=E[q]-E[w]-RP^P=e(1-b)-a-0.5s(1-b)^2\theta$. Substituting from the individual rationality and participation constraints, we have that $E[V]=e-0.5e^2-0.5s(1-b)^2\theta-0.5rb^2\theta$. The first-order condition for b is then $1-b-rb\theta+s(1-b)\theta=0$, from which it follows that $b=(1+s\theta)/(1+r\theta+s\theta)$. It is straightforward to show[1] that $\partial b/\partial\theta=-r/D<0$, $\partial b/\partial r=-(1+s\theta)\theta/D<0$, and $\partial b/\partial s=r\theta^2/D>0$, where $D=(1+r\theta+s\theta)^2$. Therefore, the optimal piece rate decreases with uncertainty (θ) and the degree of agent's risk aversion (r) and increases with the degree of principal's risk aversion (s).

(7) The optimal level of effort is given by the condition that $q'(e^*)=c'(e^*)$, which in this case implies that $e^*=1$. Given the production function $q=e$, this means that the payment is $w=0.5$ if $q\geq 1$ and $w=0$ if $q<1$. The agent's expected payoff is $E[U]=w-c(e)=w-0.5e^2$. Therefore, $E[U]=0$ if

[1] Recall that for a function $f(x)/g(x)$ we have that $[f'(x)g(x)- f(x)g'(x)]/g(x)^2$.

$e<1$ and $E[U]=0.5-0.5e^2$ if $e\geq1$. For $e\geq1$, the agent maximizes his expected payoff by choosing $e=1=e^*$. Moreover, since $E[U|e=1]>E[U|e=0]$, the agent will choose $e=1$. What remains to be seen if the agent's expected payoff evaluated at $e=1$ is greater than his outside option of 0. We have that $E[U]=0.5-0.5e^2=0.5-0.5(1^2)=0=R$. Therefore, this contract will be acceptable to the agent. On the other hand, the principal's expected payoff is $E[V]=q-w=e-w=1-0.5=0.5>0$, so the contract is acceptable to the principal as well.

(8) The three contracts can be analyzed within a general payment contract $w=a+bq$, where q is the harvest. Specifically, $w=a$ and $b=0$ for the wage labour contract, $w=a+q$ with $a<0$ and $b=1$ for the land rental, and $w=a+bq$, with $0<b<1$ for the sharecropping contract. Now, it is likely that the crop depends not only on the farmer's labour but also on other factors such as weather. Therefore, there is some uncertainty in q. When the farmer's labour can be observed, the sharing of risk is independent of the uncertainty and depends solely on the relative risk aversion of the land owner and the farmer. On the other hand, when the farmer's labour cannot be observed, b is likely to depend on the crop risk. In particular, as long as the farmer is risk averse, it will not be optimal to allocate entire risk to the farmer in the form of the land rental, and this conclusion will be more likely when there is more crop risk.