Homework 4
Piece Rate Model – Risk Aversion

Problems

In the following problems, unless otherwise stated, assume that:

- The agent's effort *e* cannot be observed or verified,
- The value of each unit of output *p* is normalized to 1,
- The agent is risk averse, and
- The principal is risk neutral.

(1)* The Sunnybrook hospital wants to employ an orthopaedic surgeon to perform knee surgeries. For each unit of effort e, the surgeon can perform 9e+u surgeries, where u is a random variable with E[u]=0 and Var[u]=4. The surgeon's cost of effort is $0.5e^2$, his outside option is $20, and his coefficient of absolute risk aversion r is 5. The hospital's outside option is $0. Design an optimal piece rate contract.

(2)* The school principal considers employing a new English teacher. For each unit of effort e, the teacher improves the standardized reading score of students by e+u, where u is a random variable with E[u]=0 and Var[u]=1. The teacher's cost of effort is $0.5e^2$, her outside option is $1, and her coefficient of absolute risk aversion r is 3. The principal's outside option is $0. Is it efficient that the principal hires this teacher?

(3)* You consider hiring a Certified Financial Analyst (CFA) to manage your investment portfolio. For each unit of effort e, the analyst increases the value of your portfolio by e+u, where u is a random variable with E[u]=0 and Var[u]=3. The analyst's cost of effort is e^2, her outside option is $0, and her coefficient of absolute risk aversion r is 3. Your outside option is $0. What is the expected value of your portfolio if you designed an optimal piece rate contract?

(4) ** For each unit of effort e, the sales associate working for Gap Inc. sells 10e+u shirts, where u is a random variable with E[u]=0 and Var[u]=3. The associate's cost of effort is $0.5e^2$, her outside option is $5, and her coefficient of absolute risk aversion r is 2. The outside option for Gap Inc. is $0. What is the associate's optimal risk premium if Gap Inc. can observe associate's effort? What is this premium if the associate's effort cannot be observed?

(5) ** You wish to hire an accountant from HR Block to help you find savings in your annual tax return. For each unit of effort e, the accountant can increase your savings by e+u, where u is a random variable with E[u]=0 and Var[u]=V. The accountant's cost of effort is $0.5e^2$ and her outside option is R, while your outside option is 0. The accountant is risk neutral, while you are risk averse with the coefficient of absolute risk aversion of

r>0. Find the optimal piece rate contract. Compare this contract to the optimal contract in the case where both you and the accountant are risk neutral.

(6) ** The principal and the agent have to decide how to split an outcome q which has a variance of V. The agent gets a share bq, while the principal gets the remaining share (1-b)q. Both the principal and the agent are risk averse, with the coefficients of absolute risk aversion of r_A and r_P, respectively. What is the value of b that optimally allocates risk between the principal and the agent (i.e. b that minimizes the sum of their risk premiums)? Using this to explain how risk should be shared in the case when the agent is risk averse and the principal is risk neutral and also in the opposite case when the agent is risk neutral and the principal is risk averse.

(7) * Consider a contract between the city of Toronto and its new mayor, where the outcome q can be interpreted as the improvement in the quality of life in the city.

a. Describe an optimal contract when the mayor's effort can be verified.

b. Describe an optimal pay for performance contract when the mayor's effort cannot be verified. Explain how this contract depends on the risk attitude of the mayor and the extent to which the mayor can control the outcome.

Solutions

(1) To find the optimal contract, we can proceed in three steps. First, we find the relationship between e and b from the incentive compatibility constraint. Specifically, the surgeon chooses e to maximize his expected utility E[U]=E[w]-c(e)-RP, where RP is the risk premium, equal to 0.5rVar[w]=$0.5b^2rV$. Substituting for what is provided in the problem, we have E[U]=a+b×9e-$0.5e^2$-$0.5b^2$(5)(4). The first-order condition for e is then 9b=e. Second, we can find the optimal piece rate by using the expected profit E[Π]=E[q]-E[w]. From the participation constraint, we have that E[w]-c(e)-RP=R, or that E[w]=R+c(e)+RP. Therefore, the expected profit can be written as E[Π]=E[q]-R-c(e)-RP=9e-20-$0.5e^2$-$0.5b^2$(5)(4)= 9e-20-$0.5e^2$-$10b^2$. In addition, substituting for e from the incentive compatibility constraint (e=9b), we have that E[Π]=9(9b)-20-0.5$(9b)^2$-$10b^2$=81b-20-$40.5b^2$-$10b^2$. The first-order condition for b is then 81-81b-20b=0, which yields b=81/101≈0.8, where ≈ indicates 'approximately equal to'. This then implies that e=9b≈7.2. Lastly, we find *a* from the surgeon's participation constraint: E[U]=R, which yields a+9be-$0.5e^2$-$10b^2$ = R. Therefore, a = R+$0.5e^2$+$10b^2$-9be = 20+0.5(7.2^2)+10(0.8^2)-9(0.8)(7.2) ≈ 20+25.92+6.4-51.84 ≈ 0.5.

(2) It is efficient to hire the teacher if and only if E[Π]+E[U]≥R+Π_R, where E[Π] and E[U] are evaluated at the optimal values for a and b. Substituting for what is provided in the problem, we have E[Π]=(1-b)e-a, E[U]=a+be-$0.5e^2$-$0.5b^2$(3)(1), R=1, and Π_R=0. Therefore, the efficiency condition simplifies to e-$0.5e^2$-$1.5b^2$≥1. To find e and b, we can follow the same steps as in Problem 1 to get e=b=0.25. Therefore, we have that e-$0.5e^2$-$1.5b^2$=0.125, which is smaller than the outside option of 1. Therefore, it is inefficient to hire the teacher.

(3) The expected value of your portfolio is E[q]=e, where the value of e depends on the contract parameters a and b. Following the same steps as in Problem 1, we have from the incentive compatibility constraint that e=0.5b and that b≈0.05. Therefore, the expected value of your portfolio is about 0.025.

(4) The risk premium is $0.5b^2rV$, which for this problem equals $0.5b^2$(2)(3)=$3b^2$. When the associate's effort can be observed, the optimal contact is a salary contract (b=0) that satisfies MB(e*)=MC(e*). Therefore, the risk premium with observable effort is 0. Intuitively, when the principal can observe the agent's effort, the contract can fully insure the agent since the impact of random factors on output can be fully distinguished from the impact of agent's effort. When the effort cannot be observed, we have to find the value for optimal b. Using the same steps as in Problem 1, we have that b≈0.94 and therefore the risk premium is $3b^2$≈2.65.

(5) This is the case where the principal is risk averse while the agent is risk neutral. The contract in this case is identical to the contract when both parties are risk neutral. To see this intuitively, remember that the piece rate of b=1 solves the incentive problem

when both parties are risk neutral. In addition, the optimal risk sharing requires that the party that is better able to handle the risk handles more risk. With the risk neutral agent and risk-averse principal, this means that the agent should completely insure the principal, or b=1. Therefore, with the risk neutral agent and risk averse principal, b=1 solves both the incentive and insurance problem. To see this formally, we can use similar steps as in problem 1. First, we find the relationship between e and b from the incentive compatibility constraint. Specifically, the accountant chooses e to maximize his expected utility E[U]=E[w]-c(e)-RP, where RP is the risk premium, equal to $0.5b^2rV=0$ since the accountant is risk neutral and therefore r=0. Substituting for what is provided in the problem, we have $E[U]=a+be-0.5e^2$. The first-order condition for e is then b=e. Second, from the participation constraint E[U]=R, we have that $a=R+0.5e^2-be$. Lastly, we can find the optimal piece rate by using the expected profit $E[\Pi]=E[q-w]-0.5r_PVar[q-w]$. Note that q-w is your net income and that the variance of this income now enters into your profit function because you are risk averse. Substituting for q and w, we have q-w=e+u-a-be-bu=(1-b)e+(1-b)u-a. Therefore, E[q-w]=(1-b)e-a and $Var[q-w]=(1-b)^2V$. Therefore, $E[\Pi]=(1-b)e-a-0.5r_P(1-b)^2V$, where $0.5r_P(1-b)^2V$ is your risk premium. Substituting for a from the participation constraint, $E[\Pi]=e-0.5e^2-0.5r_P(1-b)^2V-R$. Further substituting for e from the incentive compatibility constraint, we have $E[\Pi]=b-0.5b^2-0.5r_P(1-b)^2V-R$. The first-order condition for b is $1-b+(1-b)\ r_PV=0$, or $1+\ r_PV=b(1+\ r_PV)$, which yields b*=1. This is identical to the case where both you and your accountant are risk neutral.

(6) The principal's risk premium is $0.5r_PVar[(1-b)q]=0.5r_P(1-b)^2V$, while the agent's risk premium is $0.5r_AVar[bq]=\ 0.5r_Ab^2V$. Therefore, the total risk premium is $0.5r_P(1-b)^2V+0.5r_Ab^2V$. The problem is to minimize this total risk premium by choosing b. The first-order condition for b is $-r_P(1-b)V+r_AbV=0$, from which it follow that $b=r_P/(r_A+r_P)$. Therefore, the risk should be shared according to how risk-averse each party is. When the agent is risk averse and the principal is risk neutral ($r_A>0$ and $r_P=0$), b=0, or the principal completely insures the agent. In the opposite case, when the agent is risk neutral and the principal is risk averseal ($r_A=0$ and $r_P>0$), b=1, or the agent completely insures the principal.

(7) (**a**) When the mayor's effort could be verified, the optimal contract is a competitive fixed payment to the mayor (i.e. a payment that satisfies his participation constraint) in exchange for a level of effort that satisfies the MB(e*)=MC(e*) condition. (**b**) When the mayor's effort cannot be verified, a pay for performance type of contract may be preferred to the fixed payment contract since the contract cannot specify the effort level. Specifically, if the mayor is risk neutral, the optimal contract would specify a piece rate b=1 and would require a mayor to pay a fixed payment of –a, where a<0. If the mayor is risk averse, the optimal contract would provide the mayor with a fixed payment a>0 and a piece rate b that will be between 0 and 1. The piece rate will be closer to 0 (i.e. the contract will be more low-powered incentive contract) the more risk averse the mayor is and the less control he has over the outcome.