

Homework 2

Piece Rate Model

Problems

(1)* The Ministry of Health wants to employ a physician to treat elderly patients in a long-term care home. On average, the physician can treat 10 patients per day ($E[q]=10e$). The Ministry values the treatment of each patient at \$1. The physician's disutility of work is $0.5e^2$ and his outside option is \$25, while the Ministry's outside option is \$0. Is it efficient that the Ministry hires the physician? If so, what piece rate contract should the Ministry offer to the physician if the Ministry cannot verify how many days the physician works? Assume that both the physician and the Ministry are risk neutral.

(2)* The CIBC bank employs an in-house lawyer. On average, the lawyer successfully completes $e^{0.5}$ legal cases for each unit of effort e ($E[q]=e^{0.5}$), where each legal case is valued at \$2. The lawyer's disutility of work is $0.25e$ and her outside option is \$2, while the bank's outside option is \$1. Is it efficient that the bank employs this lawyer? If so, what piece rate contract should the bank offer to the lawyer if the lawyer's effort cannot be observed? Assume that both the lawyer and the bank are risk neutral.

(3)* Jimmy owns a car repair shop and wants to employ a new mechanic. On average, the mechanic can repair $2\ln(e)$ cars for each unit of effort e ($E[q]=2\ln(e)$), where each repair is valued at \$1. The mechanic's disutility of work is $0.25e^2$ and his outside option is \$0, while Jimmy's outside option is \$0.15. Is it efficient that Jimmy hires the mechanic? If so, what piece rate contract should Jimmy offer to the mechanic if the mechanic's effort cannot be observed? Assume that both Jimmy and the mechanic are risk neutral.

(4)** Suppose that the taxi driver has on average 10 customers for each unit of effort e ($E[q]=10e$) and that he earns \$5 for each customer. The driver's disutility of work is $0.5e^2$ and his outside option is \$250. The driver does not own the car and has to rent it from the car owner. Compare the social value of this relationship, net of any transfer between the driver and the owner, of the following two piece rate contracts: (a) the driver keeps all fares he earns, and (2) the driver shares fares equally with the owner. Assume that both the driver and the owner are risk neutral and that the driver's effort cannot be verified.

(5)** The quality of students' learning often depends not only on teacher's effort, but also on her ability to transfer knowledge. Specifically, suppose that $E[q]=e+n$, where q is the quality of students' learning, e is teacher's effort, and n is teacher's ability to transfer knowledge. Assume also that the teacher's disutility of effort is $0.5e^2$ and that

each unit of quality is valued at \$1. Explain how differences in n across teachers will affect the design of the optimal piece rate contract when the teacher's effort cannot be observed.

(6)** A sales rep is paid a fixed salary of \$20 per day if she sells 5 shirts or less and \$10 per shirt if she sells more than 5 shirts. What is the equivalent piece rate contract?

(7)** The worker produces on average one unit of output for each unit of effort ($E[q]=e$). The worker's cost of effort is $0.5e^2$. What is the efficient level of effort if the worker's employer is a monopolist in the product market such that $p=10-2q$?

(8)** Some parents tend to be late in picking their children from a day care centre. The day care centre wishes to implement a new contract with the parents that would help reduce this problem.

- a. Interpret this problem in terms of the principal-agent relationship.
- b. Discuss why the tardiness problem may arise in the first place.
- c. Discuss how a piece rate contract may solve this problem.

Solutions

(1) It is efficient for the Ministry to hire the physician if the value of the relationship evaluated at the optimal level of effort exceeds the sum of the outside options for the Ministry and the physician. That is, $pE[q]-c(e) \geq R + \Pi_R$, or $1 \times 10e^* - 0.5e^{*2} \geq 25$. The optimal level of effort e^* satisfies $pE[q'(e^*)] = c'(e^*)$, or $1 \times 10 = e^*$. This implies that $1 \times 10e^* - 0.5e^{*2} = 50$, which is greater than 25, and therefore it is optimal that the Ministry hires the physician. To find the optimal piece rate contract $[a^*, b^*]$, we start with the Ministry's expected profits $E[\Pi] = pE[q] - a - bpE[q] = 10e(1-b) - a$. In the second step, we use the physician's participation constraint (PC) to substitute for a . The PC is $E[U] = a + bpE[q] - c(e) = R$, or $a + b10e - 0.5e^2 = 25$, or $a = 25 + 0.5e^2 - 10be$. Substituting for a in the expected profits yields $E[\Pi] = 10e(1-b) - 25 - 0.5e^2 + 10be = 10e - 0.5e^2 - 25$. In the third step, we substitute for e from the incentive compatibility constraint (ICC). The ICC requires that e be such to maximize the physician's expected utility given the piece rate contract: $e = \text{argmax } E[U] = a + b10e - 0.5e^2$. The first-order condition for e yields $e = 10b$. Substituting for e in the expected profit function, we have $E[\Pi] = 10e - 0.5e^2 - 25 = 100b - 50b^2 - 25$. The first-order condition for b yields $b^* = 1$. From the ICC, we then have that $e^* = 10b^* = 10$. From the PC, we then have that $a^* = 25 + 0.5e^{*2} - 10b^*e^* = -25$. Therefore, the Ministry should offer $[a^*, b^*] = [-\$25, 1]$. This piece rate contract will achieve the optimal level of effort e^* of 10.

(2) Using the same approach as in Problem 1, we have that $e^* = 16$ and that it is efficient that the bank employs the lawyer. In addition, the optimal piece rate contract is $[a^*, b^*] = [-\$2, 1]$.

(3) Using the same approach as in Problem 1, we have that $e^* = 2$ and that it is efficient that the bank employs the lawyer. In addition, the optimal piece rate contract is $[a^*, b^*] = [-\$0.38, 1]$.

(4) The social value of the relationship net of any transfers is $pE[q] - c(e) = 10 \times 10e - 0.5e^2 = 100e - 0.5e^2$. Since the owner cannot observe the driver's effort, the driver will choose that effort that maximizes his expected utility given the piece rate contract: $E[u] = a + bpE[q] - c(e) = a + b100e - 0.5e^2$. The first-order condition for e is $100b = e$. When $b = 1$, as in the first contract, $e = 100$; when $b = 0.5$, as in the second contract, $e = 50$. Therefore, the social value is $100(100) - 0.5(100^2) = 5,000$ when $b = 1$ and $100(50) - 0.5(50^2) = 2,500$ when $b = 0.5$.

(5) Given any piece rate contract $[a, b]$, the teacher will choose e to maximize his expected utility $E[U] = a + bpE[q] - c(e) = a + b(e+n) - 0.5e^2$. The first-order condition for e yields $b = e$. Therefore, b^* is independent of n . On the other hand, a^* depends on n since from the participation constraint we have that $a = R + c(e) - bpE[q] = R + 0.5e^2 - b(e+n)$. Therefore, the variation in ability across teachers will cause a difference in the fixed component of the piece rate contract, but no difference in the variable component of the contract.

(6) The equivalent piece rate contract has the form $w=a+bpq$. Further, to induce the optimal level of effort, we need that $b=1$. In addition, for $q>5$, the commission contract and the piece rate contract are identical, so we have that $w=a+10q$. To find a , we know that $w=20$ when $q=5$, so we have that $20=a+10(5)$, from which it follows that $a=-30$. Therefore, the equivalent piece rate contract is $w=-30+10q$.

(7) Here, we have that $pq=(10-2q)q=10q-2q^2$. Therefore, $E[pq]=10e-2e^2$ and the expected marginal benefit is $10-4e$. On the other hand, the marginal cost is e . The efficient level of effort sets $10-4e^*=e^*$, which implies that $e^*=2$.

(8) (a) In this case, we can think of the day care centre as the principal and the representative parent as the agent. The output of interest, q , may be interpreted as the number of days per month that the parent is late, while e can represent an index of all types of different actions that the parent can take to ensure that he or she is on time. Other factors that are not under the parent's control can be thought of as a random variable u . (b) The problem may arise because the daycare centre cannot ascertain whether the parent is late because he or she doesn't try hard enough to be on time (e) or because of factors that the parent could not control (u). In addition, it is reasonable to assume that trying hard to be on time is costly to the parent, which alleviates the problem. (c) A piece rate contract would have a fixed and a variable component. The fixed component, a , would be negative, so we could think of it as a deposit that the parent has to pay at the beginning of each month. This deposit would then be paid back in part at the end of the month by a certain amount for each day that the parent was on time. The optimal return of deposit would balance the benefit to the centre of the parent being on time and the personal cost to the parent of being on time. The fixed payment should be low enough to induce the parent to remain with the centre and not switch to another centre that may not have such penalty for tardiness.