Appendix C
Uncertainty

Review of Statistical Concepts

The set of possible outcomes in a statistical problem is represented by a sample space *S*. A random variable *x* is a function that associates with each element *s* in *S* a real number *x*(*s*). For example, the elements of *S* may be the books in a library and *x*(*s*) may specify the number of pages in the book *s*. With the sample space comes a probability function *p* that assigns a probability *p*(*s*) to each element *s* in *S*. In our earlier example with the library books, *Prob*(*x*=100) is the probability that the book *s* has exactly 100 pages.

Each random variable has a mean denoted by $\bar{x}$, also called its expectation and denoted by E[*x*]. The formula for calculating expectations is:

$$E[x] = \sum p(s)x(s) = \bar{x}$$

Each random variable also has a variance given by

$$Var(x) = E[(x-\bar{x})^2] = \sum p(s)(x-\bar{x})^2$$

Variance is one measure of the degree of randomness of *x.*

To illustrate, consider the following example:

p(*s*)	*x*(*s*)	$p(s)x(s)$	$(x(s)-\bar{x})^2$	$p(s)(x(s)-\bar{x})^2$
1/2	0	0	4,000,000	2,000,000
1/3	3,000	1,000	1,000,000	333,333
1/6	6,000	1,000	16,000,000	2,666,667
		$\sum p(s)x(s) = \bar{x}$ =2,000		$Var(x) = \sum p(s)(x-\bar{x})^2$ =5,000,000

Given two random variables *x* and *y*, the covariance of *x* and *y* is:

$$Cov(x,y) = E[(x-\bar{x})(y-\bar{y})]$$

Covariance measures the extent to which *x* and *y* are correlated (move together).

Given two random variables *x* and *y* and two real numbers α and β, we can form a new random variable $\alpha x + \beta y$. Its expectation is given by:

$$E[\alpha x + \beta y] = \alpha E[x] + \beta E[y]$$

and its variance is equal to:

$$Var[\alpha x + \beta y] = \alpha^2 Var[x] + \beta^2 Var[y] + 2\alpha\beta Cov(x, y)$$

The two random variables *x* and *y* are statistically independent if for all numbers α and β, $\Pr ob(x = \alpha$ and $y = \beta) = \Pr ob(x = \alpha) \Pr ob(y = \beta)$. Statistical independence represents the idea that knowing the value of one of the variables provides no information about the value of the others. If *x* and *y* are statistically independent, then *Cov*(*x,y*)=0 and *Var*(*x+y*)=*Var*(*x*)+ *Var*(*y*).

Risk Premium and Certainty Equivalent

A risk-averse individual would prefer receiving a certain income of $\bar{I}$ to receiving a random income with expected value of $\bar{I}$. To illustrate, suppose that the individual preferences are given by the utility function *u*(*I*), where *I* is a random variable with expected value of $\bar{I}$. Then, for the risk-averse individual we have that:

$$u(\bar{I}) > \sum p(I)u(I) = E[u(I)]$$

The amount that the risk-averse person is willing to pay to make the switch is called the risk premium and is denoted by *RP*:

$$u(\bar{I} - RP) = E[u(I)]$$

One of the central results of decision theory under uncertainty is that the risk premium can be estimated by a simple formula:

$$RP = 0.5rVar(I)$$

This result is proved below. *r* is a parameter of the decision maker's personal preference called the coefficient of absolute risk aversion[1]. For risk-averse individuals, *r* >0 and it indicates that these individuals are willing to pay to avoid risk. For risk-neutral individuals, *r*=0. Therefore, their risk premium is also 0, which implies that they are

[1] In general, *r* may depend on $\bar{I}$, but we will assume that it does not. Intuitively, we are assuming that the degree of risk aversion does not depend on the person's income.

indifferent between receiving a certain income of $\bar{I}$ and receiving a random income with expected value of $\bar{I}$.

The amount $\bar{I} - RP$ that is left after the risk premium is paid is called the certainty equivalent and is denoted by *CE.* It is the amount of income, payable for certain, that the person regards as equivalent in value to the original, random income:

$$u(CE) = E[u(I)]$$

The advantage of focusing on the certainty equivalent is this. In general, maximizing the expected utility function $E[u(I)]$ is difficult because it depends on the whole probability distribution of income *p*(.). However, since $u(CE) = E[u(I)]$, maximizing *u*(*CE*) is equivalent to maximizing $E[u(I)]$. Moreover, if *u* is well-defined (i.e. the inverse of *u* is a positive transformation of *u*), then maximizing *CE* is the same thing as maximizing *u*(*CE*). The reason for this is that utility functions only rank outcomes and any positive transformation will not affect this ranking. Therefore, the problem of maximizing $E[u(I)]$ can be reduced to the problem of maximizing *CE*. This is obviously a much simpler problem since *CE* can be computed easily using the mean and variance of *I* as follows:

$$CE = \bar{I} - 0.5rVar(I)$$

Proof that $CE = \bar{I} - 0.5rVar(I)$

The second-order Taylor approximation to a function *u* for any point *z* is:

$$u(z) = u(\bar{x}) + (z - \bar{x})u'(\bar{x}) + 0.5(z - \bar{x})^2 u''(\bar{x})$$

where $\bar{x} = E[x]$.

Substituting *x* for *z* in this equation and computing the expectation, we have:

(1)
$$\begin{aligned} E[u(x)] &= u(\bar{x}) + E[x - \bar{x}]u'(\bar{x}) + 0.5E[(x - \bar{x})^2]u''(\bar{x}) \\ &= u(\bar{x}) + 0.5Var(x)u''(\bar{x}) \end{aligned}$$

where the second line follows since $E[x - \bar{x}]$=0 and $E[(x - \bar{x})^2] = Var(x)$.

Next, for *CE* we can use the first-order Taylor approximation (since we expect that *CE* will be close to $\bar{x}$) as follows:

(2) $$u(CE) = u(\bar{x}) + (CE - \bar{x})u'(\bar{x})$$

Lastly, we can combine equations (1) and (2) as follows:

$$
\begin{aligned}
& u(CE) = E[u(x)] \\
& \Rightarrow u(\bar{x}) + (CE - \bar{x})u'(\bar{x}) = u(\bar{x}) + 0.5Var(x)u''(\bar{x}) \\
& \Rightarrow CE = \bar{x} - 0.5\frac{u''(\bar{x})}{u''(\bar{x})}Var(x) = \bar{x} - 0.5rVar(x)
\end{aligned}
$$