Appendix A
Single Variable Optimization

The Derivative

A function *f: D*$\Rightarrow$ *R* is a rule that assigns to each number in the domain *D* a single number in the range *R*. For example, a function f(x)=2x assigns to each real number x the value of 2x.

Two useful properties of functions are:

(1) Its first derivative, denoted by f′(x)
- Represents how the function changes at point *x* if we increase *x* by a very small amount (i.e. it is the slope of the function at point *x*).
- Tells us whether the function is increasing (f′(x)>0) or decreasing (f′(x)<0) at point *x*

(2) Its second derivative, denoted by f″(x)
- Represents how the slope of the function changes at point *x* if we increase *x* by a very small amount (i.e. it is the curvature of the function at point *x*).
- Tells us whether the slope is increasing (f″(x)>0) or decreasing (f″(x)<0) at point *x*

For example, for the function f(x)=2x, the first derivative is 2 and the second derivative is 0. Therefore, the function is increasing for each x in its domain and it has a constant slope 2 that does not depend on the value of x.

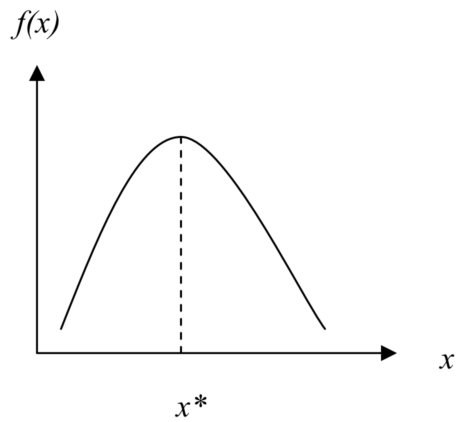
Some common formulas for derivatives are presented in the following table:

Function	Example	Derivative Rule	Derivative for Example
c (constant)	5	0	0
x^n	x^3	nx^{n-1}	$3x^{3-1}$
ln(f(x))	ln(2x)	f′(x)/f(x)	2/2x
f(x)+g(y)	$x+x^2$	f′(x)+g′(x)	1+2x
f(x)g(y)	xy^2	f′(x)g(x)+ f(x)g′(x)	y^2+2xy
f(x)/g(y)	x/y	$[f'(x)g(x)- f(x)g'(x)]/g(x)^2$	$[y-x]/y^2$
f(g(x))	$(x^2)^2$	f′(g(x))g′(x)	$2(x^2)^{2-1}2x$

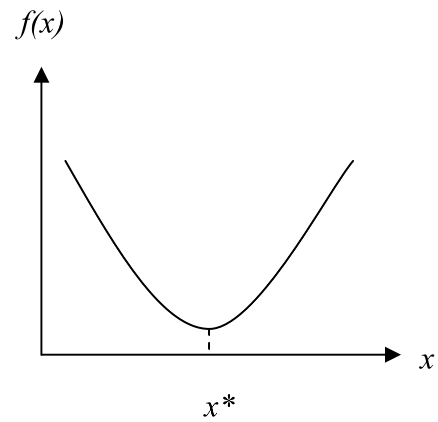
Single Variable Optimization

Consider a function *f* that depends on a single variable *x*. We say that *f* achieves a unique maximum at *x** if $f(x^*)>f(x)$ for all *x*.

Similarly, we say that *f* achieves a unique minimum at *x** if $f(x^*)<f(x)$ for all *x*.



Maximum



Minimum

When *f* is differentiable, the maximum and minimum can be defined using the first-order and second-order conditions:

	Maximum at *x**	Minimum at *x**
First-order condition:	$f'(x^*)=0$	$f'(x^*)=0$
Second-order condition:	$f''(x^*)<0$	$f''(x^*)>0$

In words, the slope of the function at both the minimum and the maximum point is zero (i.e. a tangent line to the function is flat). For the maximum, as we increase *x* the slope becomes negative; for the minimum, as we increase *x* the slope becomes positive.

For example, the function $f(x)=-x^2$ achieves its maximum at x=0 and the function $f(x)=x^2$ achieves its minimum at x=0.